Int. J. Solids Structures Vol. 16, pp. 1155–1168
Pergamon Press Ltd., 1980. Printed in Great Britain

A NEW APPROACH FOR DEVELOPING DYNAMIC THEORIES FOR STRUCTURAL ELEMENTS

PART 1: APPLICATION TO THERMOELASTIC PLATES

YALÇIN MENGI
The Middle East Technical University, Ankara, Turkey

(*Received* 30 *August* 1979)

Abstract—With the object of developing refined dynamic theories for plates, shells, beams and composites, a new technique is proposed. This technique eliminates any inconsistency between the assumed deformation or temperature shape and lateral boundary or interface conditions. Accordingly, it improves the dispersive characteristics of waves propagating in any of these structural elements. In this study the new technique is applied to thermoelastic plates. It is found that the dispersion curves predicted by the refined approximate theory duplicate very closely those derived from the exact theory without introducing any matching coefficients into the approximate theory.

INTRODUCTION

It is well known that when an approximate mathematical model governing the dynamic behavior of a continuous body with a particular geometry is developed its validity is judged usually by comparing the spectrum predicted by the model with that predicted by the exact theory or experiments. This criterion is widely used by many researchers to assist them in establishing approximate theories for plates, shells, beams or composites (see e.g. [1–11]). The procedure they use is based on a series expansion of the displacements with respect to the distance in a certain direction dictated by the geometry of the body (e.g. in the thickness direction for plates and shells, and in the lateral direction for beams). Retaining certain number of terms in the series and using a variational functional they obtain the equations of the approximate theory. To compensate the error which, they claim, is caused by truncation of series, some of the researchers introduce parameters, called matching coefficients, into the theory. They determine the values of these coefficients by adjusting certain properties of the approximate spectra to match the exact.

The work described here is initiated from the suspicion that the main source of the error in the theories mentioned above may be associated with the incompatibility between the assumed deformation shapes in these theories and the lateral boundary or interface conditions. This suspicion is based on the fact that the correct prediction of geometric dispersion by a theory depends on whether the reflection and refraction properties of the boundary and interfaces are taken into account correctly or not. In this study we present a new technique which eliminates the inconsistencies between the assumed deformation or temperature shape, and the lateral boundary or interface conditions. This technique is general in the sense that (i) it can be used to develop approximate dynamic theories for plates, shells, beams and composites; (ii) the thermal effects may be included with no difficulty; (iii) the material may be elastic or viscoelastic, isotropic or anisotropic; (iv) as many dispersion curves as desired may be included in the analysis.

In the present work, which is the first part of our study, we apply the new technique to thermoelastic, isotropic plates. In the second part the same technique is applied to thermoelastic, laminated composites. The application of the technique to beams and other composites (such as masonry wall like composites) is under study and will be reported shortly.

To develop an mth order theory for plates first we choose set of distribution functions $\{\phi_0(\bar{x}_2), \phi_1(\bar{x}_2), \ldots, \phi_m(\bar{x}_2), \phi_{m+1}(\bar{x}_2), \phi_{m+2}(\bar{x}_2)\}$, where $\bar{x}_2 = x_2/h$; x_2 is the distance measured perpendicular to the midplane of the plate; h is the half thickness of the plate. Retaining the two additional functions ϕ_{m+1} and ϕ_{m+2} in the set makes it possible to establish the constitutive relations for the face variables (the face variables are the displacements, stresses, etc. defined on the faces of the plate). These constitutive relations play a critical role in satisfying the lateral boundary conditions correctly. Using ϕ_n $(n = 0 - m)$ as weighting functions we then integrate the equations of thermoelasticity over the thickness of the plate. This gives some approximate equations expressed in terms of generalized variables (i.e. generalized displacements, stresses, etc.) and in terms of face variables. To complete the theory some additional constitutive relations are needed for the face variables and for some generalized variables whose constitutive relations are not given by the approximate equations already established. The additional relations are obtained by expanding the displacements and temperature in terms of ϕ_n $(n = 0 - (m+2))$ and by taking into account the relations between these expansions, and generalized and face variables.

In order to demonstrate the power of the technique, the flexural and longitudinal waves are studied by using first and second order theories respectively. As seen from Figs. 1 and 2 the match between the exact and approximate dispersion curves is excellent. It must be emphasized that this perfect match is obtained in spite of using lower order theories and not introducing any matching coefficients. Eliminating the need for using matching coefficients in a theory is very important because the determination of these coefficients depends on the availability of exact or experimental data and involves lengthy computations.

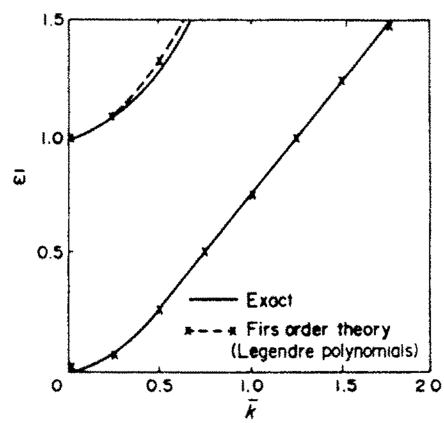

Fig. 1. Spectrum for flexural waves in a plate ($\nu = 0.25$).

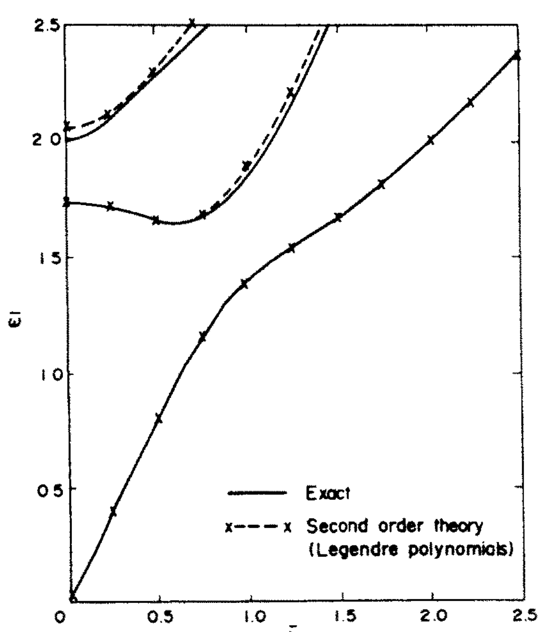

Fig. 2. Spectrum for extensional waves in a plate ($\nu = 0.25$).

INTEGRATION OF THE FIELD EQUATIONS

We assume that the plate is made of an isotropic, thermoelastic material and has a uniform thichness "$2h$". We refer the plate to a Cartesian coordinate system (x_1, x_2, x_3) in which the x_1x_3 plane coincides with the midplane of the plate. We first write the fundamental equations of linear thermoelasticity. They are

equations of motion:

$$\partial_j \tau_{ji} + f_i = \rho \ddot{u}_i \tag{1}$$

constitutive equations:

$$\tau_{ij} = \mu(\partial_i u_j + \partial_j u_i) + \delta_{ij}\lambda \partial_k u_k - \delta_{ij}\beta\theta \tag{2}$$

energy equation:

$$-\partial_k q_k + g = c_v\dot{\theta} + \beta T_0 \partial_k v_k \tag{3}$$

modified Fouries's law:

$$\tau\dot{q}_i + q_i = -k\partial_i\theta \tag{4}$$

where

- ρ mass density
- λ, μ Lamé's constants
- c_v specific heat per unit volume at constant deformation
- T_0 absolute temperature of the reference configuration
- k coefficient of heat conduction
- τ retardation time for the heat flux
- u_i displacement
- τ_{ij} stress components
- $v_i(=\dot{u}_i)$ components of particle velocity
- q_i components of heat flux
- θ temperature deviation from the reference temperature
- f_i components of body force
- g heat generation.

β is defined by $\beta = (3\lambda + 2\mu)\alpha$, where α is the coefficient of thermal expansion; δ_{ij} is the usual Kronecker delta; the dot denotes the partial differentiation with respect to time t; and ∂_i stands for $(\partial/\partial x_i)$. In writing eqns (1)–(4) the indicial notation is used. In this notation the repeated index implies summation over the range of that index. We note that the modified Fourier law, eqn (4), is obtained by adding the term "$\tau\dot{q}$" to the left hand side of the classical Fourier equation. This modification permits a finite wave speed for the thermal wave front[12].

To develop the approximate dynamic theory for plates, we start the analysis by choosing a set of distribution functions $\{\phi_n(\bar{x}_2); n = 0, 1, 2, ..\}$, where $\bar{x}_2 = x_2/h$. We assume that the ϕ_n form a complete set in the sense that a given function $f(\bar{x}_2)$ on the interval $-1 \le \bar{x}_2 \le 1$ can be represented by the series $\sum_{n=0}^{\infty} \alpha_n\phi_n(\bar{x}_2)$, i.e. $\lim_{N\to\infty} \sum_{n=0}^{N} \alpha_n\phi_n(\bar{x}_2) = f(\bar{x}_2)$, where α_n's are some constants. For developing an mth order theory we retain the elements $\{\phi_0, \phi_1, \ldots, \phi_m, \phi_{m+1}, \phi_{m+2}\}$ of the set. Further, we assume that ϕ_n $(n = 0-(m+2))$ are those elements of the set which permit us to include all the displacement or temperature distributions having the nodal points of the number from zero to $(m+2)$ along the thickness of the plate. As will be seen later, keeping the last two elements ϕ_{m+1} and ϕ_{m+2} in the set allows us to satisfy the lateral boundary conditions correctly. Without loss of generality we also assume that ϕ_n $(n = 0-(m+2))$ is an even function of $\bar{x}_2$ for n even and odd function of $\bar{x}_2$ for n odd. At this stage of the analysis we do not make any assumption regarding the orthogonality of the functions ϕ_n.

We now multiply the equations of motion by ϕ_n $(n = 0 - m)$, integrate them over the thickness and divide the resulting equations by "$2h$" (i.e. we apply the operator $(1/2h)\int_{-h}^{h}(\ )\phi_n\,dx_2$ to eqn (1)). This gives

$$\partial_1 \tau_{1i}^n + \partial_3 \tau_{3i}^n - \bar{\tau}_{2i}^n + R_i^n + f_i^n = \rho \ddot{u}_i^n \ (n = 0 - m), \tag{5}$$

where

$$(\tau_{1i}^n, \tau_{2i}^n, f_i^n, u_i^n) = \frac{1}{2h}\int_{-h}^{h} (\tau_{1i}, \tau_{2i}, f_i, u_i)\phi_n \, dx_2$$

$$\bar{\tau}_{2i}^n = \frac{1}{2h}\int_{-h}^{h} \tau_{2i} \frac{d\phi_n}{dx_2} \, dx_2$$

$$R_i^n = \frac{\phi_n(1)}{2h} \overset{*}{R}_i^n \tag{6}$$

$$\overset{*}{R}_i^n = \begin{bmatrix} R_i^- = \tau_{2i}^+ - \tau_{2i}^- \text{ for even } n \\ R_i^+ = \tau_{2i}^+ + \tau_{2i}^- \text{ for odd } n \end{bmatrix}$$

$$\tau_{2i}^{\pm} = \tau_{2i|x_2 = \pm h}.$$

To establish the constitutive relations for τ_{1i}^n, τ_{3i}^n and $\bar{\tau}_{2i}^n$, we use eqn (2). We apply the operator $(1/2h)\int_{-h}^{h}(\ )\phi_n\,dx_2$ to the constitutive equations for τ_{ii} and τ_{3i}, and the operator $(1/2h)\int_{-h}^{h}(\ )(d\phi_n/dx_2)\,dx_2$ to the constitutive equations for τ_{2i}. We thus obtain

$$\begin{aligned} \tau_{11}^n &= (2\mu + \lambda)\partial_1 u_1^n + \lambda \partial_3 u_3^n + \lambda(S_2^n - \bar{u}_2^n) - \beta\theta^n \\ \tau_{33}^n &= \lambda \partial_1 u_1^n + (2\mu + \lambda)\partial_3 u_3^n + \lambda(S_2^n - \bar{u}_2^n) - \beta\theta^n \\ \tau_{12}^n &= \mu(\partial_1 u_2^n + S_1^n - \bar{u}_1^n) \\ \tau_{32}^n &= \mu(\partial_3 u_2^n + S_3^n - \bar{u}_3^n) \\ \tau_{13}^n &= \tau_{31}^n = \mu(\partial_3 u_1^n + \partial_1 u_3^n) \ (n = 0 - m) \end{aligned} \tag{7}$$

$$\begin{aligned} \bar{\tau}_{22}^n &= \lambda \partial_1 \bar{u}_1^n + \lambda \partial_3 \bar{u}_3^n + (2\mu + \lambda)(\bar{S}_2^n - \bar{\bar{u}}_2^n) - \beta\bar{\theta}^n \\ \bar{\tau}_{21}^n &= \mu(\partial_1 \bar{u}_2^n + \bar{S}_1^n - \bar{\bar{u}}_1^n) \\ \bar{\tau}_{23}^n &= \mu(\partial_3 \bar{u}_2^n + \bar{S}_3^n - \bar{\bar{u}}_3^n) \ (n = 0 - m), \end{aligned} \tag{8}$$

where

$$\theta^n = \frac{1}{2h}\int_{-h}^{h} \theta\phi_n \, dx_2$$

$$(\bar{u}_i^n, \bar{\theta}^n) = \frac{1}{2h}\int_{-h}^{h} (u_i, \theta)\frac{d\phi_n}{dx_2} \, dx_2$$

$$\bar{\bar{u}}_i^n = \frac{1}{2h}\int_{-h}^{h} u_i \frac{d^2\phi_n}{dx_2^2} \, dx_2$$

$$S_i^n = \frac{\phi_n(1)}{2h} \overset{*}{S}_i^n \tag{9}$$

$$\bar{S}_i^n = \frac{\phi_n'(1)}{2h^2} \overset{*}{\bar{S}}_i^n$$

$$\overset{*}{S}_i^n = \begin{bmatrix} S_i^- = u_i^+ - u_i^- & \text{for even } n \\ S_i^+ = u_i^+ + u_i^- & \text{for odd } n \end{bmatrix}$$

$$\overset{*}{\bar{S}}_i^n = \begin{bmatrix} S_i^+ & \text{for even } n \\ S_i^- & \text{for odd } n \end{bmatrix}$$

$$u_i^{\mp} = u_i|_{x_2 = \mp h}.$$

In the fifth of eqns (9) the prime denotes the derivative with respect to the nondimensional distance $\bar{x}_2$, i.e. $\phi_n' = (\mathrm{d}\phi_n/\mathrm{d}\bar{x}_2)$.

We now apply the operator $(1/2h)\int_{-h}^{h}(\ )\phi_n\,\mathrm{d}x_2\ (n = 0 - m)$ to the energy equation, eqn (3), to get

$$\begin{aligned} &-(\partial_1 q_1^n + \partial_3 q_3^n - \bar{q}_2^n + Q^n) + g^n \\ &= c_v\dot{\theta}^n + \beta T_0(\partial_1 v_1^n + \partial_3 v_3^n - \bar{v}_2^n + \dot{S}_2^n)(n = 0 - m), \end{aligned} \tag{10}$$

where $v_i^n = \dot{u}_i^n$, $\bar{v}_i^n = \dot{\bar{u}}_i^n$ and

$$\begin{aligned} (q_1^n, q_3^n, g^n) &= \frac{1}{2h}\int_{-h}^{h}(q_1, q_3, g)\phi_n\,\mathrm{d}x_2 \\ \bar{q}_2^n &= \frac{1}{2h}\int_{-h}^{h} q_2\frac{\mathrm{d}\phi_n}{\mathrm{d}x_2}\,\mathrm{d}x_2 \\ Q^n &= \frac{\phi_n(1)}{2h}\overset{*}{Q}^n \\ \overset{*}{Q}^n &= \begin{bmatrix} Q^- = q_2^+ - q_2^- & \text{for even } n \\ Q^+ = q_2^+ + q_2^- & \text{for odd } n \end{bmatrix} \\ q_2^{\mp} &= q_2|_{x_2 = \mp h}. \end{aligned} \tag{11}$$

The integrated form of the Fourier equations for q_1^n, q_2^n and $\bar{q}_2^n$ can be found by using eqn (4). To this end we apply the operator $(1/2h)\int_{-h}^{h}(\ )\phi_n\,\mathrm{d}x_2$ to the Fourier equations for q_1 and q_3, and the operator $(1/2h)\int_{-h}^{h}(\ )(\mathrm{d}\phi_n/\mathrm{d}x_2)\,\mathrm{d}x_2$ to the Fourier equation for q_2. We obtain

$$\begin{aligned} \tau\dot{q}_1^n + q_1^n &= -k\partial_1\theta^n \\ \tau\dot{q}_3^n + q_3^n &= -k\partial_3\theta^n \quad (n = 0 - m) \end{aligned} \tag{12}$$

$$\tau\dot{\bar{q}}_2^n + \bar{q}_2^n = -k(\bar{\psi}^n - \bar{\bar{\theta}}^n) \quad (n = 0 - m), \tag{13}$$

where

$$\begin{aligned} \bar{\bar{\theta}}^n &= \frac{1}{2h}\int_{-h}^{h}\theta\frac{\mathrm{d}^2\phi_n}{\mathrm{d}x_2^2}\,\mathrm{d}x_2 \\ \bar{\psi}^n &= \frac{\phi_n'(1)}{2h^2}\overset{*}{\psi}^n \\ \overset{*}{\psi} &= \begin{bmatrix} \psi^+ = \theta^+ + \theta^- & \text{for even } n \\ \psi^- = \theta^+ - \theta^- & \text{for odd } n \end{bmatrix} \\ \theta^{\mp} &= \theta|_{x_2 = \mp h}. \end{aligned} \tag{14}$$

The integration of the field equations is now complete. Equations (5), (7), (8), (10), (12) and (13) constitute $16(m+1)$ equations. Stress or displacement, and temperature or heat flux boundary conditions on the lateral surfaces of the plate give another eight equations which can be expressed in terms of the face variables $S_i^{\mp}$, $R_i^{\mp}$, $\psi^{\mp}$, $Q^{\mp}$. In fact, on the faces of the plate, as exact boundary conditions, we specify quantities composed of one member of each of the pairs

$$(\tau_{2i}^{+}, u_i^{+});\quad (\tau_{2i}^{-}, u_i^{-})\quad (i = l-3)$$
$$(q_2^{+}, \theta^{+});\quad (q_2^{-}, \theta^{-}).$$

In the approximate theory these eight boundary conditions can be expressed in terms of face variables using eqns $(6)_4$, $(9)_6$, $(11)_4$ and $(14)_3$. They take the following form. Quantities composed of one member of each of the pairs

$$\left(\frac{R_i^{+}+R_i^{-}}{2}, \frac{S_i^{+}+S_i^{-}}{2}\right);\quad \left(\frac{R_i^{+}-R_i^{-}}{2}, \frac{S_i^{+}-S_i^{-}}{2}\right)\quad (i = l-3)$$
$$\left(\frac{Q^{+}+Q^{-}}{2}, \frac{\psi^{+}+\psi^{-}}{2}\right);\quad \left(\frac{Q^{+}-Q^{-}}{2}, \frac{\psi^{+}-\psi^{-}}{2}\right)$$

are specified on the lateral surfaces. Thus the number of available equations is $[16(m+1)+8]$. On the other hand, the number of unknowns (τ_{1i}^{n}, τ_{3i}^{n}, $\bar{\tau}_{2i}^{n}$, u_i^{n}, $\bar{u}_i^{n}$, $\bar{\bar{u}}_i^{n}$, q_1^{n}, q_3^{n}, $\bar{q}_2^{n}$, θ^{n}, $\bar{\theta}^{n}$, $\bar{\bar{\theta}}^{n}$, $R_i^{\mp}$, $S_i^{\mp}$, $Q^{\mp}$, $\psi^{\mp}$) is $[24(m+1)+16]$. Therefore, to complete our mathematical model we need $[8(m+1)+8]$ more equations.

ADDITIONAL EQUATIONS

The $[8(m+1)+8]$ additional equations can be obtained by establishing the constitutive relations for the face variables $S_i^{\mp}$, $\psi^{\mp}$ and for the generalized variables $\bar{u}_i^{n}$, $\bar{\bar{u}}_i^{n}$, $\bar{\theta}^{n}$, $\bar{\bar{\theta}}^{n}$. To this end we expand the displacements u_i and the temperature θ in terms of ϕ_n $(n = 0-(m+2))$:

$$u_i = \sum_{k=0}^{m+2} a_k^{i}\phi_k(\bar{x}_2) \tag{15}$$

$$\theta = \sum_{k=0}^{m+2} b_k\phi_k(\bar{x}_2), \tag{16}$$

where the coefficients a_k^{i} and b_k are the functions of x_1, x_3 and t. Using eqns (2) and (15), for τ_{2i} we get

$$\tau_{2i} = \frac{\mu}{h}\sum_{k=0}^{m+2} a_k^{i}\phi_k' + \mu\partial_i u_2 \quad \text{for } i = l, 3$$
$$\tau_{22} = \frac{(2\mu+\lambda)}{h}\sum_{k=0}^{m+2} a_k^{2}\phi_k' + \lambda(\partial_1 u_1 + \partial_3 u_3) - \beta\theta. \tag{17}$$

When we insert eqn (16) into the Fourier law eqn (4) with $i = 2$ we obtain

$$\tau\dot{q}_2 + q_2 = \frac{-k}{h}\sum_{k=0}^{m+2} b_k\phi_k'. \tag{18}$$

Substitution of eqns (15) and (16) into the expressions defining $S_i^{\mp}$ (eqn 9_6), $\bar{u}_i^{n}$ (eqn 9_2), $\bar{\bar{u}}_i^{n}$ (eqn 9_3), $\psi^{\mp}$ (eqn 14_3), $\bar{\theta}^{n}$ (eqn 9_2) and $\bar{\bar{\theta}}^{n}$ (eqn 14_1) give

$$S_i^{+} = 2\sum_{k=0,2,\ldots}^{p+2} a_k^{i}\phi_k(1)$$

$$S_i^{-} = 2\sum_{k=1,3,\ldots}^{p'+2} a_k^{i}\phi_k(1)$$

$$\bar{u}_i^{n} = \begin{cases} \displaystyle\sum_{k=1,3,\ldots}^{p'+2} \bar{c}_{nk}a_k^{i} & \text{for even } n \\ \displaystyle\sum_{k=0,2,\ldots}^{p+2} \bar{c}_{nk}a_k^{i} & \text{for odd } n \end{cases} \tag{19}$$

$$\bar{\bar{u}}_i^n = \begin{cases} \sum_{k=0,2,\ldots}^{p+2} \bar{\bar{c}}_{nk} a_k^i & \text{for even } n \\ \sum_{k=1,3,\ldots}^{p'+2} \bar{\bar{c}}_{nk} a_k^i & \text{for odd } n \end{cases}$$

and

$$\psi^+ = 2 \sum_{k=0,2}^{p+2} b_k \phi_k(1)$$

$$\psi^- = 2 \sum_{k=1,3,\ldots}^{p'+2} b_k \phi_k(1)$$

$$\bar{\theta}^n = \begin{cases} \sum_{k=1,3,\ldots}^{p'+2} \bar{c}_{nk} b_k & \text{for even } n \\ \sum_{k=0,2,\ldots}^{p+2} \bar{c}_{nk} b_k & \text{for odd } n \end{cases} \tag{20}$$

$$\bar{\bar{\theta}}^n = \begin{cases} \sum_{k=0,2,\ldots}^{p+2} \bar{\bar{c}}_{nk} b_k & \text{for even } n \\ \sum_{k=1,3,\ldots}^{p'+2} \bar{\bar{c}}_{nk} b_k & \text{for odd } n, \end{cases}$$

where

$$\begin{aligned} p = m; \quad & p' = m - 1 \quad \text{for even } m \\ p = m - 1; \quad & p' = m \quad \text{for odd } m \end{aligned} \tag{21}$$

and

$$\begin{aligned} \bar{c}_{nk} &= \frac{1}{2h} \int_{-1}^{+1} \phi_n' \phi_k \, d\bar{x}_2 \\ \bar{\bar{c}}_{nk} &= \frac{1}{2h^2} \int_{-1}^{+1} \phi_n'' \phi_k \, d\bar{x}_2. \end{aligned} \tag{22}$$

To complete the development of the additional equations it remains to determine the coefficients a_k^i and b_k appearing in eqns (19) and (20). For this purpose we apply the operator $(1/2h) \int_{-h}^{+h} () \phi_n \, dx_2$ $(n = 0 - m)$ to eqns (15) and (16); and we write eqns (17) and (18) at $x_2 = \mp h$, and we add and subtract them. This gives set of algebraic equations governing the a_k^i and b_k. They are

equations governing a_k^i with even k:

$$\begin{aligned} \sum_{k=0,2,\ldots}^{p+2} c_{nk} a_k^i &= u_i^n \quad (n = 0, 2, \ldots, p) \\ \sum_{k=0,2,\ldots}^{p+2} \phi_k'(1) a_k^i &= A_i^+ \end{aligned} \tag{23}$$

equations governing a_k^i with odd k:

$$\begin{aligned} \sum_{k=1,3,\ldots}^{p'+2} c_{nk} a_k^i &= u_i^n \quad (n = 1, 3, \ldots, p') \\ \sum_{k=1,3,\ldots}^{p'+2} \phi_k'(1) a_k^i &= A_i^- \end{aligned} \tag{24}$$

equations governing b_k with even k:

$$\sum_{k=0,2,\ldots}^{p+2} c_{nk} b_k = \theta^n \quad (n = 0, 2, \ldots, p)$$

$$\sum_{k=0,2,\ldots}^{p+2} \phi_k'(1) b_k = B^+ \tag{25}$$

equations governing b_k with odd k:

$$\sum_{k=1,3,\ldots}^{p'+2} c_{nk} b_k = \theta^n \quad (n = 1, 3, \ldots, p')$$

$$\sum_{k=1,3,\ldots}^{p'+2} \phi_k'(1) b_k = B^-, \tag{26}$$

where

$$A_i^{\mp} = \begin{cases} \dfrac{h}{2\mu} R_i^{\pm} - \dfrac{h}{2}\partial_i S_2^{\pm} \text{ for } i = 1, 3 \\ \dfrac{h}{2(2\mu+\lambda)} R_i^{\pm} - \dfrac{h}{2}\dfrac{\lambda}{(2\mu+\lambda)}(\partial_1 S_1^{\pm} + \partial_3 S_3^{\pm}) + \dfrac{h}{2}\dfrac{\beta}{(2\mu+\lambda)}\psi^{\pm} \text{ for } i = 2 \end{cases} \tag{27}$$

$$B^{\mp} = -\frac{h}{2k}(\tau \dot{Q}^{\pm} + Q^{\pm}) \tag{28}$$

$$c_{nk} = \frac{1}{2}\int_{-1}^{1} \phi_n \phi_k \, \mathrm{d}\bar{x}_2. \tag{29}$$

It should be noted that in the derivation of eqns (23)–(26) the expressions defining $R_i^{\mp}$, $S_i^{\mp}$, $Q_i^{\mp}$ and $\psi^{\mp}$ are used.

When we substitute the a_k^i and b_k determined from eqns (23)–(26) into eqns (19) and (20) we obtain the equations of the form

$$S_i^+ = 2\Big(\sum_{k=0,2,\ldots}^{p} \gamma_k u_i^k + \gamma^+ A_i^+\Big)$$

$$S_i^- = 2\Big(\sum_{k=1,3,\ldots}^{p'} \gamma_k u_i^k + \gamma^- A_i^-\Big) \tag{30}$$

$$\bar{u}_i^n = \begin{cases} \sum_{k=1,3,\ldots}^{p'} c'_{nk} u_i^k + c_n'^{+} A_i^- & \text{for even } n \\ \sum_{k=0,2,\ldots}^{p} c'_{nk} u_i^k + c_n'^{-} A_i^+ & \text{for odd } n \end{cases}$$

$$\bar{\bar{u}}_i^n = \begin{cases} \sum_{k=0,2,\ldots}^{p} c''_{nk} u_i^k + c_n''^{+} A_i^+ & \text{for even } n \\ \sum_{k=1,3,\ldots}^{p'} c''_{nk} u_i^k + c_n''^{-} A_i^- & \text{for odd } n \end{cases}$$

and

$$\psi^+ = 2\Big(\sum_{k=0,2,\ldots}^{p} \gamma_k \theta^k + \gamma^+ B^+\Big)$$

$$\psi^- = 2\Big(\sum_{k=1,3,\ldots}^{p'} \gamma_k \theta^k + \gamma^- B^-\Big)$$

$$\bar{\theta}^n = \begin{cases} \sum_{k=1,3,\ldots}^{p'} c'_{nk}\theta^k + c'^{+}_n B^- & \text{for even } n \\ \sum_{k=0,2,\ldots}^{p} c'_{nk}\theta^k + c'^{-}_n B^+ & \text{for odd } n \end{cases} \tag{31}$$

$$\bar{\bar{\theta}} = \begin{cases} \sum_{k=0,2,\ldots}^{p} c''_{nk}\theta^k + c''^{+}_n B^+ & \text{for even } n \\ \sum_{k=1,3,\ldots}^{p'} c''_{nk}\theta^k + c''^{-}_n B^- & \text{for odd } n. \end{cases}$$

In eqns (30) and (31) γ_k, c'_{nk}, c''_{nk}, $\gamma^{\mp}$, $c'^{\mp}_n$ and $c''^{\mp}_n$ are some constants whose explicit values can be determined using eqns (19)–(29) whenever the functions ϕ_n are specified.

The $[8(m+1)+8]$ equations in eqns (30) and (31) and the equations derived in the previous section constitute the governing equations of our approximate theory. We note that eqns $(30)_1$, $(30)_2$, $(31)_1$ and $(31)_2$ represent the constitutive relations for the face variables. The derivation of these equations is based on the expansions, eqns (15) and (16) and the field equations of thermoelasticity. The use of these equations together with the lateral boundary conditions permits us to satisfy the lateral boundary conditions correctly and eliminates any inconsistency between the assumed displacement or temperature shape and the lateral boundary conditions.

THE CASE OF ORTHOGANAL ϕ_n's

The constitutive relations, eqns (30) and (31), hold whether the ϕ_n are orthogonal or not. However, the computation of the constants appearing in these equations will be simpler if an orthogonal set of functions is chosen. In what follows we will present the expressions defining these constants when the ϕ_n form an orthogonal set.

In accordance with the orthogonality of the ϕ_n, we write first, using eqn (29):

$$c_{nk} = \frac{1}{2}\int_{-1}^{+1} \phi_n\phi_k \, d\bar{x}_2 = c_{\underline{n}}\delta_{\underline{n}k}, \tag{32}$$

where c_n is a constant whose value will be known whenever the ϕ_n are chosen. In eqn (32), the underlined repeated index implies that there is no summation over that index. When eqns (23)–(26) are solved for a_k^i and b_k with the aid of eqn (32) we get

$$\begin{aligned} a_n^i &= \frac{u_i^{\underline{n}}}{c_{\underline{n}}} \quad (n = 0, 1, \ldots, m) \\ a^i_{p+2} &= -\frac{1}{\phi'_{p+2}(1)}\left(\sum_{k=0,2,\ldots}^{p} \phi'_k(1)\frac{u_i^k}{c_k} - A_i^+\right) \\ a^i_{p'+2} &= -\frac{1}{\phi'_{p'+2}(1)}\left(\sum_{k=1,3,\ldots}^{p'} \phi'_k(1)\frac{u_i^k}{c_k} - A_i^-\right) \end{aligned} \tag{33}$$

and

$$\begin{aligned} b_n &= \frac{\theta^{\underline{n}}}{c_{\underline{n}}} \quad (n = 0, 1, \ldots, m) \\ b_{p+2} &= -\frac{1}{\phi'_{p+2}(1)}\left(\sum_{k=0,2,\ldots}^{p} \phi'_k(1)\frac{\theta^k}{c_k} - B^+\right) \\ b_{p'+2} &= -\frac{1}{\phi'_{p'+2}(1)}\left(\sum_{k=1,3,\ldots}^{p'} \phi'_k(1)\frac{\theta^k}{x_k} - B^-\right). \end{aligned} \tag{34}$$

Substitution of eqns (33) and (34) into eqns (19) and (20) and comparison of the resulting equations with eqns (30) and (31) give

$$
\gamma_k = \begin{cases} \dfrac{1}{c_{\underline{k}}}(\phi_{\underline{k}}(1) - \gamma^{+}\phi'_{\underline{k}}(1)) & (k = 0, 2, \ldots, p) \\ \dfrac{1}{c_{\underline{k}}}(\phi_{\underline{k}}(1) - \gamma^{-}\phi'_{\underline{k}}(1)) & (k = 1, 3, \ldots, p') \end{cases} \tag{35}
$$

$$
\gamma^{+} = \frac{\phi_{p+2}(1)}{\phi'_{p+2}(1)}; \quad \gamma^{-} = \frac{\phi_{p'+2}(1)}{\phi'_{p'+2}(1)},
$$

$$
c'_{nk} = \begin{cases} \dfrac{1}{c_{\underline{k}}}(\bar{c}_{n\underline{k}} - c'^{+}_{n}\phi'_{\underline{k}}(1)) & \text{for even } n \; (k = 1, 3, \ldots, p') \\ \dfrac{1}{c_{\underline{k}}}(\bar{c}_{n\underline{k}} - c'^{-}_{n}\phi'_{\underline{k}}(1)) & \text{for odd } n \; (k = 0, 2, \ldots, p) \end{cases} \tag{36}
$$

$$
c'^{+}_{n} = \frac{\bar{c}_{n,p'+2}}{\phi'_{p'+2}(1)} \quad \text{for even } n
$$

$$
c'^{-}_{n} = \frac{\bar{c}_{n,p+2}}{\phi'_{p+2}(1)} \quad \text{for odd } n,
$$

and

$$
c''_{nk} = \begin{cases} \dfrac{1}{c_{\underline{k}}}(\bar{\bar{c}}_{n\underline{k}} - c''^{+}_{n}\phi'_{\underline{k}}(1)) & \text{for even } n \; (k = 0, 2, \ldots, p) \\ \dfrac{1}{c_{\underline{k}}}(\bar{\bar{c}}_{n\underline{k}} - c''^{-}_{n}\phi'_{\underline{k}}(1)) & \text{for odd } n \; (k = 1, 3, \ldots, p') \end{cases} \tag{37}
$$

$$
c''^{+}_{n} = \frac{\bar{\bar{c}}_{n,p+2}}{\phi'_{p+2}(1)} \quad \text{for even } n
$$

$$
c''^{-}_{n} = \frac{\bar{\bar{c}}_{n,p'+2}}{\phi'_{p'+2}(1)} \quad \text{for odd } n.
$$

THE VALUES OF CONSTANTS FOR LEGENDRE POLYNOMIALS

Using the formulas obtained in this section, tables for the values of the constants can be prepared for various selected ϕ_n and for various orders of the approximate theory. As an illustration we present here the values of constants when

$$
\phi_n(\bar{x}_2) = P_n(\bar{x}_2),
$$

where P_n is the nth order Legendre polynomial, and for orders of the approximate theory corresponding to $m = 0$, 1 and 2. Since Legendre polymonials are orthogonal, the constants are computed by using the formulas in eqns (36) and (37), and are presented in Table 1. The dash in the table indicates that the corresponding constant does not exist in the approximate theory.

EXAMPLES

To show the power of the present approximate theory, we now compare dispersion curves predicted by the approximate and exact theories. We make this comparison for flexural and longitudinal waves and by using lower order approximate theories, namely, first order for the flexural and second order for the longitudinal waves. We use lower order theories because we know that the match between approximate and exact spectral lines will improve as the order of the approximate theory increases. Our choosing the order of the approximate theory for longitudinal waves as two, rather than one, is motivated by the strong coupling between the second and third modes (which correspond to thickness shear and stretch modes) of these waves. We now neglect the thermal effects and proceed to derive the frequency equations for flexural and longitudinal waves. In our analysis we assume that the plate faces are free of traction and, we choose the ϕ_n to be Legendre polynomials.

FLEXURAL WAVES

If the symmetry of u_2 and antisymmetry of u_1 with respect to the midplane of the plate are taken into account for the flexural waves in x_1 direction, the equations of the first order approximate theory reduce to

equations of motion:

$$\partial_1\tau_{12}^0 - \bar{\tau}_{22}^0 = \rho\ddot{u}_2^{\,0}$$
$$\partial_1\tau_{11}^1 - \bar{\tau}_{21}^1 = \rho\ddot{u}_1^{\,1} \tag{38}$$

constitutive equations:

$$\tau_{12}^0 = \mu(\partial_1 u_2^{\,0} + S_1^{\,0} - \bar{u}_1^{\,0})$$
$$\bar{\tau}_{22}^0 = \lambda\partial_1\bar{u}_1^{\,0} + (2\mu+\lambda)(\bar{S}_2^{\,0} - \bar{\bar{u}}_2^{\,0}) \tag{39}$$
$$\tau_{11}^1 = (2\mu+\lambda)\partial_1 u_1^{\,1} + \lambda(S_2^{\,1} - \bar{u}_2^{\,1})$$
$$\bar{\tau}_{21}^1 = \mu(\partial_1\bar{u}_2^{\,1} + \bar{S}_1^{\,1} - \bar{\bar{u}}_1^{\,1}),$$

where

$$S_1^{\,0} = \frac{\phi_0(1)}{2h}S_1^- = \frac{1}{2h}S_1^-$$
$$S_2^{\,1} = \frac{\phi_1(1)}{2h}S_2^+ = \frac{1}{2h}S_2^+$$
$$\bar{S}_2^{\,0} = \frac{\phi_0'(1)}{2h^2}S_2^+ = 0$$
$$\bar{S}_1^{\,1} = \frac{\phi_1'(1)}{2h^2}S_1^- = \frac{1}{2h^2}S_1^-, \tag{40}$$

and

additional equations:

$$S_1^- = 2(\gamma_1 u_1^{\,1} + \gamma^- A_1^-)$$
$$S_2^+ = 2(\gamma_0 u_2^{\,0} + \gamma^+ A_2^+)$$
$$\bar{u}_2^{\,1} = c'_{10}u_2^{\,0}$$
$$\bar{u}_1^{\,0} = \bar{\bar{u}}_1^{\,1} = \bar{\bar{u}}_2^{\,0} = 0, \tag{41}$$

where

$$A_1^- = -\frac{h}{2}\partial_1 S_2^+$$
$$A_2^+ = -\frac{h}{2}\frac{\lambda}{(2\mu+\lambda)}\partial_1 S_1^-. \tag{42}$$

The coefficients in eqns (41) are given in Table 1 and in writing these equations the zero coefficients are taken into account. Equations (42) are obtained by using eqns (27) and traction free lateral boundary conditions.

To obtain the frequency equation we let the trial solution for the unknown variables be of the form: constant times $e^{i(kx_1-\omega t)}$, where k is the wave number and ω is the frequency, and substitute it into the governing equations, eqns (38)–(42). Then the condition for having a

Table 1. Values of constants for zeroth, first and second order theories

const / order	γ_0	γ_1	γ_2	γ^+	γ^-	c'_{01}	c'^+_0	c'_{21}	c'^+_2	c'_{10}
0	1	-	-	1/3	1	-	0	-	-	-
1	1	5/2	-	1/3	1/6	0	0	-	-	1/h
2	1	5/2	7/2	1/10	1/6	0	0	3/h	0	1/h
const / order	c'_{12}	c'^-_1	c''_{00}	c''_{02}	c''^+_0	c''_{20}	c''_{22}	c''^+_2	c''_{11}	c''^-_1
0	-	-	0	-	0	-	-	-	-	-
1	-	0	0	-	0	-	-	-	0	0
2	0	0	0	0	0	$3/h^2$	0	0	0	0

nontrivial solution gives the frequency equation which, after some manipulation, takes the form

$$\begin{vmatrix} \rho\omega^2-\mu k^2 & 0 & \frac{\mu}{2h}(ik) & 0 \\ -c'_{10}(\lambda+\mu)(ik) & \rho\omega^2-(2\mu+\lambda)k^2 & -\frac{\mu}{2h^2} & \frac{\lambda}{2h}(ik) \\ 2\gamma_0 & 0 & -h\gamma^+\xi(ik) & -1 \\ 0 & 2\gamma_1 & -1 & -h\gamma^-(ik) \end{vmatrix} = 0, \tag{43}$$

where $\xi = \dfrac{\lambda}{2\mu+\lambda}$.

LONGITUDINAL WAVES

For longitudinal waves in x_1 direction, u_1 and u_2 have respectively symmetric and antisymmetric distributions with respect to the midplane of the plate. In view of this property of displacement components, the equations of the second order approximate theory reduce to

equations of motion:

$$\begin{aligned} \partial_1\tau^k_{11} - \bar{\tau}^k_{21} &= \rho\ddot{u}_1{}^k \quad (k=0,2) \\ \partial_1\tau^1_{12} - \bar{\tau}^1_{22} &= \rho\ddot{u}_2{}^1 \end{aligned} \tag{44}$$

constitutive equations:

$$\begin{aligned} \tau^k_{11} &= (2\mu+\lambda)\partial_1 u_1{}^k + \lambda(S_2{}^k - \bar{u}_2{}^k) \quad (k=0,2) \\ \bar{\tau}^k_{21} &= \mu(\partial_1\bar{u}_2{}^k + \bar{S}_1{}^k - \bar{u}_1{}^k) \quad (k=0,2) \\ \tau^1_{12} &= \mu(\partial_1 u_2{}^1 + S_1{}^1 - \bar{u}_1{}^1) \\ \bar{\tau}^1_{22} &= \lambda\partial_1\bar{u}_1{}^1 + (2\mu+\lambda)(\bar{S}_2{}^1 - \bar{u}_2{}^1), \end{aligned} \tag{45}$$

where

$$S_2{}^0 = \frac{\phi_0(1)}{2h}S_2{}^- = \frac{1}{2h}S_2{}^-$$

$$S_1{}^1 = \frac{\phi_1(1)}{2h}S_1{}^+ = \frac{1}{2h}S_1{}^+$$

$$S_2{}^2 = \frac{\phi_2(1)}{2h}S_2{}^- = \frac{1}{2h}S_2{}^-$$

$$\begin{aligned}\bar{S}_1{}^0 &= \frac{\phi_0'(1)}{2h^2} S_1{}^+ = 0\\ \bar{S}_2{}^1 &= \frac{\phi_1'(1)}{2h^2} S_2{}^- = \frac{1}{2h^2} S_2{}^-\\ \bar{S}_1{}^2 &= \frac{\phi_2'(1)}{2h^2} S_1{}^+ = \frac{3}{2h^2} S_1{}^+\end{aligned} \tag{46}$$

and

additional equations:

$$\begin{aligned}&S_1{}^+ = 2(\gamma_0 u_1{}^0 + \gamma_2 u_1{}^2 + \gamma^+ A_1{}^+)\\ &S_2{}^- = 2(\gamma_1 u_2{}^1 + \gamma^- A_2{}^-)\\ &\bar{u}_1{}^1 = c_{10}' u_1{}^0;\quad \bar{u}_2{}^2 = c_{21}' u_2{}^1;\quad \bar{\bar{u}}_1{}^2 = c_{20}'' u_1{}^0\\ &\bar{u}_2{}^0 = \bar{\bar{u}}_1{}^0 = \bar{\bar{u}}_2{}^1 = 0,\end{aligned} \tag{47}$$

where

$$\begin{aligned}A_1{}^+ &= -\frac{h}{2}\partial_1 S_2{}^-\\ A_2{}^- &= -\frac{h}{2}\xi\partial_1 S_1{}^+.\end{aligned} \tag{48}$$

The frequency equation for longitudinal waves can be obtained by using eqns (44)–(48). It is

$$\begin{vmatrix} \rho\omega^2-(2\mu+\lambda)k^2 & 0 & 0 & \frac{\lambda}{2h}(ik) & 0\\ -c_{10}'(\lambda+\mu)(ik) & \rho\omega^2-\mu k^2 & 0 & -\frac{2\mu+\lambda}{2h^2} & \frac{\mu}{2h}(ik)\\ c_{20}''\mu & -c_{21}'(\lambda+\mu)(ik) & \rho\omega^2-(2\mu+\lambda)k^2 & \frac{\lambda}{2h}(ik) & -\frac{3\mu}{2h^2}\\ 2\gamma_0 & 0 & 2\gamma_2 & -h\gamma^+(ik) & -1\\ 0 & 2\gamma_1 & 0 & -1 & -h\gamma^-\xi(ik) \end{vmatrix} = 0. \tag{49}$$

NUMERICAL RESULTS

For the value of Poisson's ratio $\nu = 0.25$ the approximate dispersion curves for flexural and longitudinal waves are obtained using eqns (43) and (49) respectively, and are compared with those derived from the exact theory in Figs. 1 and 2. In these figures $\bar{k}$ and $\bar{\omega}$ are nondimensional wave number and frequency defined by $\bar{k} = (2h/\pi)k$ and $\bar{\omega} = (2h/\pi c_s)\omega$, where $c_s = \surd(\mu/\rho)$ is the shear wave velocity. As seen from the figures the approximate and exact spectral lines match very closely. We note that there is a small difference in the third cut-by frequency of the longitudinal waves. However, we expect that this difference will disappear if the order of the approximate theory is increased.

The excellent match in the figures is obtained without using matching coefficients. Avoiding the use of matching coefficients in developing an approximate theory is very important because the determination of matching coefficients depends on the availability of exact or experimental data and involves lengthy computations.

Note—After the present work was submitted for publication an important study on plates by R. D. Mindlin titled "Vibrations of Quartz Plates with HP-67 Pocket Calculator" appeared in Computers and Structures (Vol. 10, pp. 751–759, 1979). In this study Mindlin studied the natural frequencies of a quartz plate of finite length by expanding the displacements in series of exact characteristic functions of the infinite plate. The technique proposed by Mindlin is however completely different than that used in the present study.

Acknowledgements—The author wishes to acknowledge the support provided by Scientific Affairs Division of NATO (Research Grant No: 1446). This paper is dedicated to my teacher Hugh D. McNiven.

REFERENCES

1. R. D. Mindlin, Influence on rotary inertia and shear on flexural vibrations of isotropic, elastic plates. *J. Appl. Mech.* **18**, 31 (1951).
2. E. Reissner, The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.* **12**, 69 (1945).
3. T. R. Kane and R. D. Mindlin, High-frequency extensional vibrations of plates. *J. Appl. Mech.* **23**, 277 (1956).
4. R. D. Mindlin and M. A. Medick, Extensional vibrations of elastic plates. *J. Appl. Mech.* **26**, 561 (1959).
5. P. C. Y. Lee and Z. Nikodem, An approximate theory for high-frequency vibrations of elastic plates. *Int. J. Solids Structures* **8**, 581 (1972).
6. S. Timoshenko, On transverse vibrations of bars of uniform cross-section. *Phil. Mag.* **43**, 125 (1922).
7. R. D. Mindlin and H. D. McNiven, Axially symmetric waves in elastic rods. *J. Appl. Mech.* **27**, 145 (1960).
8. Y. Mengi and H. D. McNiven, Axially symmetric waves in transversely isotropic rods. *J. Acoust. Soc. Am.* **50**, 248 (1971).
9. M. C. Dökmeci, An isothermal theory of anisotropic rods. *J. Engng Mathematics*, **9**, 311 (1975).
10. C. T. Sun, J. D. Achenbach and G. Herrmann, Continuum theory for a laminated medium. *J. Appl. Mech.* **35**, 467 (1968).
11. A. Bedford, D. S. Drumheller and H. J. Sutherland, On modelling the dynamics of composite materials. *Mechanics Today* (Edited by S. Nemat-Nasser), Vol. 3, p. 1. Pergamon, Oxford (1976).
12. J. D. Achebach, The influence of heat conduction on propagating stress jumps. *J. Mech. Phys. Solids* **16**, 273 (1968).